Derivatives of Inverse Trig Functions

-We've learned before that we can find the inverse of a graph by reflecting the graph across the line $y = x$

-If we have a differentiable function with a tangent at the point $\left(a, f\left(a\right)\right)$, then the reflection will produce a tangent at the point $\left(f\left(a\right), a\right)$.

-This means the slope of the tangent on f will be a reciprocal f^{-1}.

Theorem

-If f is differentiable at every point on the interval I and $\frac{df}{dx}$ is never zero on I, then f has an inverse and f^{-1} is differentiable at every point on the interval $f\left(I\right)$

Derivative of Arcsine

-We know $x = \sin y$ is differentiable on the open interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and the derivative cosine is positive there.

-Our Theorem assures that the inverse function $y = \sin^{-1}\left(x\right)$ is differentiable through the interval $-1 < x < 1$.

-It can't be differentiated at -1 or 1 because it would be a vertical tangent.

-We find $y = \sin^{-1}\left(x\right)$ as follows:

$$y = \sin^{-1}\left(x\right)$$

$$\sin y = x$$

$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

-This is safe because $\cos y \neq 0$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

-We can replace $\cos y$ with $\sqrt{1-(\sin y)^2}$ which is $\sqrt{1-x^2}$

Thus,

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

-If u is a differentiable function of x with $|u| < 1$

$$\frac{d}{dx}\left(\sin^{-1}(u)\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Example

Find $\frac{d}{dx}\left(\sin^{-1}(x^2)\right)$

$$= \frac{1}{\sqrt{1-\left(x^2\right)^2}} \bullet \frac{d}{dx}\left(x^2\right)$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

Derivative of Arctangent

-Although the function $y = \sin^{-1}\left(x\right)$ has a rather narrow domain of $\left[-1,1\right]$, the function $y = \tan^{-1}\left(x\right)$ is defined for all real numbers.

$$\frac{d}{dx}\tan^{-1}\left(u\right) = \frac{1}{1+u^2}\frac{du}{dx}$$

Example-Moving Particle

-A particle move along the x-axis so that its position at any time $t \geq 0$ is $x\left(t\right) = \tan^{-1}\left(\sqrt{t}\right)$. What is the velocity when $t = 16$?

$$v\left(t\right) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1+\left(\sqrt{t}\right)^2} \bullet \frac{d}{dt}\left(t\right)$$

$$v\left(t\right)\frac{1}{1+t} \bullet \frac{1}{2\sqrt{t}}$$

$$v\left(16\right) = \frac{1}{1+16} \bullet \frac{1}{2\sqrt{16}} = \frac{1}{136}$$

Derivative of Arcsecant

$$\frac{d}{dx}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Example

Find $\frac{d}{dx}\sec^{-1}\left(5x^4\right)$

$$= \frac{1}{|5x^4|\sqrt{\left(5x^4\right)^2-1}} \bullet \frac{d}{dx}\left(5x^4\right)$$

$$= \frac{1}{5x^4\sqrt{25x^8-1}} \bullet 20x^3$$

$$= \frac{4}{5\sqrt{25x^8-1}}$$

Inverse Function-Inverse Cofunction Identities

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

$$\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x$$

Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$